NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2667

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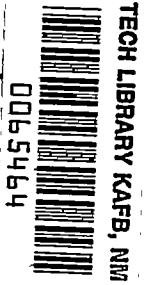
Brown University

NACA

Washington

March 1952

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SUMMARY

A basic theory of generalized linearized supersonic conical flow for both inside and outside the Mach cone was developed and applied to several specific problems including unsteady-flow conditions. A triangular lifting wing in pitching and rolling with both subsonic and supersonic leading edges was investigated and pressure coefficients were obtained. A family of thin sweptback triangular wings having symmetrical thickness distribution was also investigated and analytic expressions for wave drag and pressure coefficients were determined. Values of wave drag coefficients were calculated and the results presented graphically. This theory stems from a fundamental idea of Mr. G. N. Ward.

INTRODUCTION

Of the methods and theories of linearized supersonic flow one of the most productive of results directly applicable to aerodynamic problems has been the theory of conical flow. Conical flow refers to flow in which the pressure perturbation and velocity components are constant along straight lines or rays passing through a fixed point; for such flow the assumption of linearity is not needed and nonlinear conical flows are of great practical interest. In the generalization of conical flow presented in this report, however, the linearizing assumptions are necessary and the usual wave equation of steady linearized supersonic flow will be considered.

In generalizing the concept of conical flow attention is focused on the homogeneity of the solutions with the vertex as origin. In regular conical flow the solutions for the velocity components are homogeneous of degree 0, while the corresponding velocity potential whose gradient is the vector velocity is homogeneous of degree 1. In generalized conical flow solutions may be considered for which the velocity potential is homogeneous of degree n with the velocity components homogeneous of degree $n - 1$. The quantity n, called the degree of the generalized conical flow, may usually be an integer but does not need to be. A further generalization comes from superposition of solutions of different homogeneity.

It is the purpose of this report to present the basic theory of generalized linearized conical flow as developed from a fundamental idea of Ward and to illustrate the theory by some aerodynamic applications. The particular elements of linearized supersonic-flow theory which will be needed are presented in brief. The point of view used in the analysis is that used in reference 1, which thus may be considered a basic reference for this report. Some of the essential steps in the development are the same as those in the conical-flow theory of Goldstein and Ward (reference 2). For a careful and thorough investigation and discussion of the conical-flow theory the reader is also referred to the work of Lagerstrom (reference 3). For an investigation of generalized conical-flow theory from quite a different point of view reference may be made to the theory of Germain (reference 4). No attempt is made to give more than the most recent references as quite complete bibliographies may be found in the references which are cited.

The coordinate system used is the right-handed system used in references 1 to 3 with the third coordinate in the free-stream direction. There are several advantages which may be claimed for this particular orientation: The timelike position in the wave equation is delegated to the last coordinate; in accordance with customary notation with this equation; the variables x and y appear on an equal basis; and with x taken as the lateral variable a flat wing appears horizontal on an x-y projection. The principal disadvantage is that the first coordinate does not appear in two-dimensional equations. The investigation reported herein was conducted at Brown University under the sponsorship and with the financial assistance of the National Advisory Committee for Aeronautics.

SYMBOLS

φ	velocity potential
∇	operator
a	velocity of sound
p	pressure
ρ	density
S	entropy
x,y,z	Cartesian coordinates
M	Mach number
V	free-stream velocity

u, v, w	perturbation velocity components
α	angle of attack or of local slope
γ	ratio of specific heats
C_L	lift coefficient
C_D	drag coefficient
m	oblique transformation parameter
R	hyperbolic distance
J	Jacobian
τ	general homogeneous wave solution
r, θ, z	cylindrical coordinates
t	basic conical coordinate
s, ϵ, ζ	conical coordinates within Mach cone
$\sigma, \theta_1, \theta_2$	conical coordinates outside Mach cone
f, g	arbitrary functions
$\alpha(\tau), \beta(\tau), \gamma(\tau)$	homogeneous functions
ξ	fundamental variable
Rl	real part
p	rolling angular velocity
q	pitching angular velocity
s	variable for Jacobi's imaginary transformation
c_0	root chord
t_1, t_2, s	geometrical wing parameters
T	root thickness ratio
ψ	thickness function

Φ   velocity potential in unsteady flow

t   time

O   operator involving solution of wave equation

ω   frequency

k   reduced frequency

h(R)   function of hyperbolic distance

n   homogeneity parameter

Γ   gamma function

J   .   Bessel function

ELEMENTS OF LINEARIZED SUPERSONIC FLOW

Basic Flow Equations

With the usual assumptions of zero viscosity and heat conduction,
initial irrotationality and isentropy, uniform stagnation enthalpy and
fluid composition, and no body forces, it may be shown that the fluid is
always irrotational and isentropic and that there exists a velocity
potential φ whose gradient is the vector velocity of the flow field.
For steady flow this quantity satisfies the equation

$$a^2 \nabla^2 \varphi = (\nabla \varphi) \nabla \left(\frac{1}{2} \nabla \varphi^2 \right) \tag{1}$$

where a is the local speed of sound in the medium, defined as

$$a^2 = \left(\frac{\partial p}{\partial \rho} \right)_S \tag{2}$$

If the flow consists of a uniform flow of velocity V with velocity
of sound a_0 which is slightly perturbed the potential equation may be
expressed approximately, with terms of an order higher than O in the
perturbation dropped, in the form

$$\varphi_{xx} + \varphi_{yy} + (1 - M^2)\varphi_{zz} = 0 \tag{3}$$

where

$$M = V/a_o \tag{4}$$

Two principal assumptions are necessary for the validity of this linearized potential equation. These are:

(1) The lateral velocity perturbation components must be small compared with a_o. This requires that the quantity αM be small compared with unity where α is the local inclination of the flow at any point. Thus flows with small values of α are not necessarily linearized for large M in the hypersonic range.

(2) The axial velocity perturbation component must be small compared with $|V - a_o|$. This condition would most immediately be expressible in terms of restrictions on the pressure perturbation. However, boundary conditions are usually set in terms of flow inclination, and the appropriate condition which appears from the transonic similitude theory is that $\dfrac{\alpha}{|M^2 - 1|^{3/2}}\left(\dfrac{\gamma + 1}{2}\right)$ be small compared with unity.

With such singularities as occur at subsonic leading edges or vortex sheet edges one or both of these assumptions fail locally.

Except in the incompressible or low-speed range $M \ll 1$ these two assumptions require automatically that the flow inclination α itself be small compared with unity, permitting the boundary conditions to be linearized. For a surface locally parallel to the x-axis the boundary condition would be

$$\varphi_y = v = V\alpha \tag{5}$$

where α here is the local inclination of the surface to the principal flow direction. With flow in the low-speed range the linearized potential equation is valid with nonlinear or unlinearized boundary conditions.

The linearized potential equation is considered to be an equation for the perturbation potential. The three Cartesian derivatives of this potential are the three perturbation velocity components u, v, and w. The perturbation pressure, or difference between the local pressure and that in the free stream, is expressed

$$p = -\rho\left(Vw + \frac{u^2 + v^2}{2}\right) \tag{6}$$

The terms in u and v, although nonlinear, are properly included and
necessary in general; the skeptic may attempt to satisfy the condition
of zero normal pressure gradient on a circular cone at zero incidence
without them. However, for many cases of importance and, indeed, for
all those exemplified in this report the additional "planar-system" or
"mean-surface" assumption holds, and the pressure terms in u and v
are properly dropped leaving the simplified pressure equation

$$p = -\rho V w \tag{7}$$

This additional assumption is discussed below.

Mean-Surface Assumption

The useful additional restriction here discussed has been previously
made by most investigators under the terminology of "planar system,"
"quasi cylinder," "flat body," "mean surface," and others. In its most
general form it is assumed that the boundary conditions on a body may
be satisfied at appropriate points on a mean surface. This mean surface
must be a part of a general cylinder with z-axis directrix; that is,
must be everywhere parallel to the z-axis. In order for the assumption
to be valid it is necessary that the body closely approximate the mean
surface and that the inclination of the body surface to the mean surface
be small. The case where the body is a wing and the mean surface is the
xz-plane is the one most commonly encountered.

The importance of this additional assumption is that it gives a
direct correspondence between the superposition of solutions with the
same mean surface and the corresponding body shapes. The assumption
also insures the validity of the simplified pressure equation (7).

Prandtl-Glauert Transformation

If with the axial variables unchanged the lateral variables are
transformed by

$$x' = \sqrt{M^2 - 1}\; x \tag{8a}$$

$$y' = \sqrt{M^2 - 1}\; y \tag{8b}$$

the potential equation becomes

$$\varphi_{x'x'} + \varphi_{y'y'} - \varphi_{zz} = 0 \tag{9}$$

which is of the same form as equation (3) with the Mach number equal to $\sqrt{2}$. Where the planar-system or mean-surface assumption is applicable it is permissible to set

$$\varphi' = \sqrt{M^2 - 1}\ \varphi \tag{10}$$

this relation leaving the angle of attack, thickness ratio, and so forth unchanged under the transformation. However, there must be a change in aspect ratio, gap ratio, and so forth

$$R' = \sqrt{M^2 - 1}\ R \tag{11}$$

and there is a change in the appropriate lift and drag coefficients

$$C_L' = \sqrt{M^2 - 1}\ C_L \tag{12a}$$

$$C_D' = \sqrt{M^2 - 1}\ C_D \tag{12b}$$

In the general case where the mean-surface assumption is invalid the factor in equation (10) must be $(M^2 - 1)$, with the same factor appearing in equation (12a), and the factor $(M^2 - 1)^{3/2}$, in equation (12b). The angle of attack and thickness ratio are changed by the factor $\sqrt{M^2 - 1}$.

In the analysis henceforth it is assumed that the Mach number equals $\sqrt{2}$ so that the primes in equation (9) are dropped. Reduction of the results to other values of the Mach number is carried out by the equations given above.

Oblique Transformation

The useful oblique transformation is essentially a Lorentz transformation and was first applied to steady supersonic flow by Jones (reference 5). The transformation is in the independent variables

$$x = \frac{1}{\sqrt{1 - m^2}}(x' - mz') \tag{13a}$$

$$z = \frac{1}{\sqrt{1 - m^2}}(z' - mx') \tag{13b}$$

$$y = y' \tag{13c}$$

These transformations form an Abelian group with the inverse transformation obtained by changing the sign of m. The Cartesian velocity components obey the inverse transformation law

$$u = \frac{1}{\sqrt{1 - m^2}}(u' + mw') \tag{14a}$$

$$w = \frac{1}{\sqrt{1 - m^2}}(w' + mu') \tag{14b}$$

$$v = v' \tag{14c}$$

Three other properties of this transformation are the invariance of the hyperbolic distance, the Jacobian of the transformation being unity, and the homographic form of the transformation for x/z:

$$R = \sqrt{z^2 - x^2 - y^2} = \sqrt{(z')^2 - (x')^2 - (y')^2} \tag{15a}$$

$$J\left(\frac{x,y,z}{x',y',z'}\right) = J\left(\frac{x,z}{x',z'}\right) = 1 \tag{15b}$$

$$\left(\frac{x}{z}\right) = \frac{\left(\frac{x'}{z'}\right) - m}{1 - m\left(\frac{x'}{z'}\right)} \tag{15c}$$

Symmetry Considerations

Where the mean surface is a plane and a planar system exists it is convenient to separate the potential with regard to symmetry in the y-direction

$$\varphi = \varphi_s + \varphi_a \qquad (16a)$$

$$\varphi_s = \frac{1}{2}\left[\varphi(y) + \varphi(-y)\right] \qquad (16b)$$

$$\varphi_a = \frac{1}{2}\left[\varphi(y) - \varphi(-y)\right] \qquad (16c)$$

The symmetric part corresponds to the thickness distribution of the body under consideration; the antisymmetric part, to the mean camber and lift distribution. Both φ_s and φ_a satisfy the potential equation individually and it is possible to treat the two parts separately and independently.

Where the mean surface is symmetrical with respect to the x-axis considerations of lateral symmetry may be applied. With a laterally symmetric thickness or camber distribution the potential and pressure solutions are laterally symmetric; with a laterally antisymmetric camber distribution an antisymmetric lift distribution results.

Supersonic and Subsonic Edges

The edges of a mean-surface plan form are conveniently classified according to the local-flow orientation. The free-stream velocity may be considered to be composed of two perpendicular velocity components, a normal one perpendicular to the edge and a tangential one parallel to the edge. The normal one, taken positive from the region off the plan form onto the plan form, is divided by the speed of sound in the free stream to give a normal Mach number M_n. If this quantity is positive the edge is termed leading; if negative, trailing. If its magnitude is greater than 1, the edge is supersonic; if less than 1, subsonic. If $M_n = 0$ the edge may be called a (subsonic) side edge.

The nature of the edge as indicated by this classification strongly affects the nature of solutions for the flow field. The Kutta condition preventing an infinite behavior of lifting solutions must be applied on all subsonic trailing edges.

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SOLUTIONS HOMOGENEOUS OF DEGREE 0

As a preliminary to expressing the generalized conical-flow solution the properties of solutions to the wave equation which are homogeneous of degree 0 are studied. Denoting the quantity of interest by τ, with a change to cylindrical coordinates, the wave equation may be expressed

$$\tau_{xx} + \tau_{yy} - \tau_{zz} = 0 \tag{17a}$$

$$\tau_{rr} + \frac{1}{r}\tau_r + \frac{1}{r^2}\tau_{\theta\theta} - \tau_{zz} = 0 \tag{17b}$$

With the new variable

$$t = r/z$$

introduced to replace r, the function τ must be a function of t and θ alone. The equation for τ becomes

$$t^2(1 - t^2)\tau_{tt} + t(1 - 2t^2)\tau_t + \tau_{\theta\theta} = 0 \tag{18}$$

In a similar manner the homogeneous solutions to the equation of characteristics

$$\tau_x{}^2 + \tau_y{}^2 - \tau_z{}^2 = 0 \tag{19}$$

satisfy the equation of characteristics of equation (18)

$$t^2(1 - t^2)\tau_t{}^2 + \tau_\theta{}^2 = 0 \tag{20}$$

The interior and exterior of the Mach cone are treated separately.

Mach cone interior.- In the case of the Mach cone interior the coordinate

$$s = \text{sech}^{-1}t$$

is introduced and the wave and characteristic equations become

$$\tau_{ss} + \tau_{\theta\theta} = 0 \tag{22a}$$

$$(\tau_s - i\tau_\theta)(\tau_s + i\tau_\theta) = 0 \tag{22b}$$

Introducing the complex variable ϵ of the original Busemann linearized conical-flow theory

$$\epsilon = \exp(s + i\theta) \tag{23}$$

the solutions to equations (22) may be expressed

$$\tau = f(\epsilon) + g(\epsilon^*) \tag{24a}$$

$$\tau = f(\epsilon) \quad \text{or} \quad g(\epsilon^*) \tag{24b}$$

where f and g are arbitrary analytic functions. Of interest for the present purpose are the particular functions ϵ, ϵ^*, ζ, and ζ^*, where

$$\zeta = \frac{2\epsilon}{1 + \epsilon^2} \tag{25}$$

The principal reason for the use of this function is that for $\theta = 0$ it is real and equal to t and for $t = 1$ it is real and equal to $\sec\theta$; this gives it the nature of a physical coordinate for a flat wing.

Mach cone exterior.- In the case of the Mach cone exterior the following coordinate is introduced

$$\sigma = \sec^{-1}t \tag{26}$$

The wave and characteristic equations take the form

$$\tau_{\sigma\sigma} - \tau_{\theta\theta} = 0 \tag{27a}$$

$$\tau_\sigma^2 - \tau_\theta^2 = 0 \tag{27b}$$

Introducing the characteristic coordinates

$$\theta_1 = \theta + \sigma \qquad (28a)$$

$$\theta_2 = \theta - \sigma \qquad (28b)$$

the solutions may be expressed

$$\tau = f(\theta_1) + g(\theta_2) \qquad (29a)$$

$$\tau = f(\theta_1) \quad \text{or} \quad g(\theta_2) \qquad (29b)$$

The particular solutions of interest are simply θ_1 and θ_2.

The Cartesian derivatives of τ are functions which are homogeneous of degree -1. Functions proportional to these Cartesian derivatives are needed which are functions of τ alone and hence homogeneous of degree 0

$$\frac{\alpha(\tau)}{\tau_x} = \frac{\beta(\tau)}{\tau_y} = \frac{\gamma(\tau)}{\tau_z} \qquad (30)$$

Such functions, provided they exist, are not unique but may be made definite by arbitrarily establishing one of them. It is convenient to do this by setting

$$\gamma(\tau) \equiv 1 \qquad (31)$$

In finding these functions it is not necessary to investigate ϵ^* or ζ^* because the desired functions will be the complex conjugates of the corresponding functions for ϵ and ζ. Although a change from ϵ to ϵ^*, for example, does make a difference in the functions, a change from ϵ to $\tau = f(\epsilon)$ will make no essential difference, the formulation in terms of the different independent variables being, of course, changed.

Carrying out the indicated Cartesian differentiations yields, after some analysis, the following results:

τ	α	β	γ
ϵ	$-\dfrac{1+\epsilon^2}{2\epsilon}$	$-i\,\dfrac{1-\epsilon^2}{2\epsilon}$	1
ζ	$-\dfrac{1}{\zeta}$	$-i\,\dfrac{\sqrt{1-\zeta^2}}{\zeta}$	1
θ_1	$-\cos\theta_1$	$-\sin\theta_1$	1
θ_2	$-\cos\theta_2$	$-\sin\theta_2$	1

On the Mach cone, $\epsilon = e^{i\theta}$, $\zeta = \sec\theta$, $\theta_1 = \theta_2 = \theta$, $\alpha = -\cos\theta$, and $\beta = -\sin\theta$ in all cases.

GENERALIZED CONICAL FLOW

Fundamental Solutions

The general solution will be established as a function of the "solution point" (x_0,y_0,z_0) in terms of an integral taken over values of the "argument point" (x,y,z). The designations τ_0 and τ are used for the values of the homogeneous quantity of interest τ at the solution and argument points, respectively. A new basic variable is defined as a function of these points which vanishes when they coincide:

$$\xi = x_0\alpha(\tau) + y_0\beta(\tau) + z_0\gamma(\tau) \tag{32a}$$

$$\xi = 0 \qquad \text{for} \quad \tau = \tau_0 \tag{32b}$$

where the variable vanishes as a result of Euler's formula for homogeneous functions of degree 0

$$x\tau_x + y\tau_y + z\tau_z = 0 \tag{33}$$

and equation (3b). The fundamental idea of Mr. G. N. Ward was the definition and use of this new basic variable.

The generalized conical solution may now be expressed

$$\varphi(x_0, y_0, z_0) = \int^{\tau_0} g(\xi, \tau) \, d\tau \qquad (34)$$

where the absence of the lower limit indicates that the integral is an indefinite one with the usual arbitrary constant. In the exterior of the Mach cone the general solution is the superposition of solutions of the form of equation (34) with θ_1 and θ_2 chosen for τ. In the interior a real solution is obtained by taking the real part of a solution with τ equal to ϵ or ζ, amounting to a superposition of a solution (equation (34)) with its complex conjugate solution.

That equation (34) does give a solution of the wave equation is easily ascertained by carrying out the indicated differentiations and using the properties of equations (17a), (19), (30), (32a), and (32b). The essential restrictions on g are that it have first and second derivatives with respect to ξ and that it be analytic in τ within the Mach cone. The velocity components are expressed

$$u_0 = \varphi_{x_0} = \tau_{0_{x_0}} \, g(0, \tau_0) + \int^{\tau_0} \alpha(\tau) \, \frac{\partial g(\xi, \tau)}{\partial \xi} \, d\tau \qquad (35a)$$

$$v_0 = \varphi_{y_0} = \tau_{0_{y_0}} \, g(0, \tau_0) + \int^{\tau_0} \beta(\tau) \, \frac{\partial g(\xi, \tau)}{\partial \xi} \, d\tau \qquad (35b)$$

$$w_0 = \varphi_{z_0} = \tau_{0_{z_0}} \, g(0, \tau_0) + \int^{\tau_0} \gamma(\tau) \, \frac{\partial g(\xi, \tau)}{\partial \xi} \, d\tau \qquad (35c)$$

For most cases of physical interest $g(0, \tau) = 0$ and the first terms (homogeneous of degree -1) in equations (35) do not appear.

The solution procedure is divided into three parts: First, the solution upstream from the vertex exterior to the Mach cone, if there is any, is to be found, then the solution exterior to the Mach cone downstream from the vertex, and finally the solution within the downstream Mach cone. In each case the solution found in the previous step provides boundary conditions for the part of the problem at hand. The solution in the upstream Mach cone is assumed null.

Oblique Transformation

The oblique transformation is applied by equating a known solution of the wave equation in the coordinates x, y, and z to a solution in the coordinates x', y', and z'. This gives a new solution expressed in terms of the primed coordinates

$$\varphi(x,y,z) = \varphi'(x',y',z') \tag{36}$$

The behavior of the various basic variables under the transformation is first obtained. The variables ζ, $\cos\theta$, and $\cos\theta_2$ all obey the same transformation law as does $t\cos\theta = x/z$ in equation (15c). Thus,

$$\zeta = \frac{\zeta' - m}{1 - m\zeta'} \tag{37a}$$

$$\cos\theta_1 = \frac{\cos\theta_1' - m}{1 - m\cos\theta_1'} \tag{37b}$$

$$\cos\theta_2 = \frac{\cos\theta_2' - m}{1 - m\cos\theta_2'} \tag{37c}$$

The quantity ϵ also follows a homographic transformation, but one which is the square root of the other

$$\epsilon = \frac{\epsilon' - \dfrac{m}{1 + \sqrt{1 - m^2}}}{1 - \dfrac{m}{1 + \sqrt{1 - m^2}}\,\epsilon'} \tag{37d}$$

The quantities α and β follow the same transformation laws in all cases provided that the relation $\gamma = 1$ is maintained

$$\alpha = \frac{\alpha' + m}{1 + m\alpha'} \tag{38a}$$

$$\beta = \frac{\sqrt{1 - m^2}}{1 + m\alpha'} \beta' \tag{38b}$$

With these results it is easy to show that

$$\xi = \frac{\sqrt{1 - m^2}}{1 + m\alpha'} \xi' \tag{38c}$$

where the ξ' is defined in the same way in terms of the primed coordinates, with $\gamma' = 1$.

The solution in the primed coordinates

$$\varphi'(x_0', y_0', z_0') = \int^{\tau_0'} g'(\xi', \tau') \, d\tau' \tag{39}$$

may now be expressed in terms of the function $\tau(\tau')$ expressed by equations (37), and relation (36). The desired relation is given by

$$g'(\xi', \tau') = g \left[\frac{\sqrt{1 - m^2}}{1 + m\alpha'} \xi', \tau(\tau') \right] \frac{d\tau(\tau')}{d\tau'} \tag{40}$$

Equations (35) connecting the velocity components may be used as a check.

Conical Flow

The linearized conical-flow theory is obtained immediately by setting

$$g(\xi, \tau) = \xi \frac{df(\tau)}{d\tau} \tag{41}$$

This leads to the usual expressions for the interior of the Mach cone found by Busemann

$$w = R\ell \left[\underline{f}(\epsilon) \right] \tag{42a}$$

$$u = R\ell\left(-\int \frac{1 + \epsilon^2}{2\epsilon}\, df\right) \tag{42b}$$

$$v = R\ell\left(-i\int \frac{1 - \epsilon^2}{2\epsilon}\, df\right) \tag{42c}$$

or the analogous expressions in ζ

$$w = R\ell\left[\overline{f(\zeta)}\right] \tag{43a}$$

$$u = R\ell\left(-\int \frac{1}{\zeta}\, df\right) \tag{43b}$$

$$v = R\ell\left(-i\int \frac{\sqrt{1 - \zeta^2}}{\zeta}\, df\right) \tag{43c}$$

For the exterior of the Mach cone

$$w = f_1(\theta_1) + f_2(\theta_2) \tag{44a}$$

$$u = -\int \cos\theta_1\, df_1 - \int \cos\theta_2\, df_2 \tag{44b}$$

$$v = -\int \sin\theta_1\, df_1 - \int \sin\theta_2\, df_2 \tag{44c}$$

Thus the well-known linearized supersonic conical-flow theory appears as a special case of the theory here presented.

Second-Degree Conical Flow

The case which is of most interest here is that for which the potential is homogeneous of degree 2, with

$$g(\xi,\tau) = \frac{\xi^2}{2} \frac{df(\tau)}{d\tau} \qquad (45)$$

this corresponding to

$$\varphi_{zz} = Rl\, f(\tau) \qquad (46)$$

Noting on the plane $y = 0$ that

$$\xi = z_0 - \frac{x_0}{\zeta} = z_0\left(1 - \frac{t_0}{\zeta}\right) \qquad (47)$$

the velocity components may be expressed in the interior of the Mach cone and on $y_0 = 0$ as

$$w = z_0 Rl\left[\int\left(1 - \frac{t_0}{\zeta}\right) df(\zeta)\right] \qquad (48a)$$

$$u = z_0 Rl\left[-\int \frac{1}{\zeta}\left(1 - \frac{t_0}{\zeta}\right) df\right] \qquad (48b)$$

$$v = z_0 Rl\left[-i\int \frac{\sqrt{1 - \zeta^2}}{\zeta}\left(1 - \frac{t_0}{\zeta}\right) df\right] \qquad (48c)$$

In the exterior of the Mach cone, again on the plane $y_0 = 0$, the analogous expressions are

$$w = z_0\left[\int (1 - t_0 \cos \theta_1)\, df_1(\theta_1) + \int (1 - t_0 \cos \theta_2)\, df_2(\theta_2)\right] \qquad (49a)$$

$$u = z_0\left[-\int \cos \theta_1 (1 - t_0 \cos \theta_1)\, df_1 - \int \cos \theta_2 (1 - t_0 \cos \theta_2)\, df_2\right] \qquad (49b)$$

$$v = z_0\left[-\int \sin \theta_1 (1 - t_0 \cos \theta_1)\, df_1 - \int \sin \theta_2 (1 - t_0 \cos \theta_2)\, df_2\right] \qquad (49c)$$

As is the case with problems in conical flow, the procedure is to find a function f so that the prescribed boundary conditions are satisfied. The quadratures corresponding to the relations given above are needed to complete the problem. The examples worked out below serve to illustrate the method.

ROLLING AND PITCHING WINGS

General Considerations

Wings of triangular shape in steady rolling and pitching motion are now considered, with the apex of the triangle leading and with a straight supersonic trailing edge. Only the cases with lateral symmetry are considered but cases of yawed wings may be obtained by a suitable use of the oblique transformation. Such solutions are antisymmetric in the sense of the section titled "Symmetry Considerations." For the wing in reversed flow the methods of this report do not work. However, the pitching- and rolling-moment damping coefficients of the original wing apply for the reversed wing, as shown by the reversed-flow theorems of Brown (reference 6) and, in still more general form, of Ursell and Ward (reference 7).

In order for the steady-state equations to be valid the motion must be steady in that the local angle of attack at a point on the wing is constant. For pitching motion this requires that the flight path be curvilinear; for rolling motion, that the rolling angular velocity be constant. In these cases the problem is identical with a problem in direct flight with the angle of attack proportional to z or x.

Where these conditions are not met but where the second derivatives with respect to time may be neglected the quasi-steady theory may be used. In the cases here discussed the analysis is similar to that of the steady case and the same functions appear.

Triangular Wings within Mach Cone

The choice of the function $f(\zeta)$ in the cases involving a triangular wing with lateral symmetry within the Mach cone is made through the following considerations:

(1) The function f may have at worst a $-3/2$-power singularity at $\zeta = \pm t_1$, the leading edges of the wing

(2) The function $df/d\zeta$ must have at least a double zero at $\zeta = 0$ in order that the integrals for u and v converge

(3) The function f must be bounded for large values of ζ, corresponding to the intersection of the Mach cone with the plane of the wing

(4) Lateral-symmetry considerations require that f be an even function of ζ in the pitching case and an odd one in the rolling case

The functions are thus expressed, apart from a multiplicative constant, as

$$f_q(\zeta) = A_q \; \frac{\zeta^2 - \frac{2}{3} t_1{}^2}{\left(t_1{}^2 - \zeta^2\right)^{3/2}} \tag{50}$$

in the pitching case and

$$f_p(\zeta) = A_p \; \frac{\frac{1}{3} \zeta^3}{\left(t_1{}^2 - \zeta^2\right)^{3/2}} \tag{51}$$

in the rolling case. The remainder of the problem consists of the carrying out of the appropriate quadratures in order to evaluate the arbitrary constants in terms of the physical problem and to obtain the desired velocity component or pressure solutions.

The quadratures needed to establish the arbitrary constants are the appropriate terms from the expression

$$v(x_0, 0, z_0) = Rl \left[-i \int^{y_0 = 0} \left(z_0 - \frac{x_0}{\zeta}\right) \frac{\sqrt{1 - \zeta^2}}{\zeta} \, df \right] \tag{52}$$

or in terms of the pitching and rolling angular velocities

$$q = -\frac{v_q}{z_0} = Rl \left[iA_q \int_{i\infty}^{0} \frac{\zeta^2 \sqrt{1 - \zeta^2}}{\left(t_1{}^2 - \zeta^2\right)^{5/2}} \, d\zeta \right] \tag{53a}$$

$$p = -\frac{v_p}{x_0} = -Rl \left[iA_p t_1{}^2 \int_{i\infty}^{0} \frac{\sqrt{1 - \zeta^2}}{\left(t_1{}^2 - \zeta^2\right)^{5/2}} \, d\zeta \right] \tag{53b}$$

These integrals are evaluated by means of Jacobi's imaginary transformation

$$\zeta = \frac{i\sqrt{1 - s^2}}{s} \tag{54}$$

in terms of which the integrals become

$$q = -A_q \int_0^1 \frac{\sqrt{1 - s^2} \, ds}{\left[1 - (t_1')^2 s^2\right]^{5/2}} \tag{55a}$$

$$p = -A_p t_1^2 \int_0^1 \frac{s^2 \, ds}{\sqrt{1 - s^2}\left[1 - (t_1')^2 s^2\right]^{5/2}} \tag{55b}$$

where

$$(t_1')^2 = 1 - t_1^2 \tag{56}$$

These integrals may be expressed through the classical, although lengthy, methods of elliptic integrals. The resulting expressions are

$$A_q = \frac{-3q t_1^2 (1 - t_1^2)}{(1 - 2t_1^2)E'(t_1) + t_1^2 K'(t_1)} \tag{57a}$$

$$A_p = \frac{-3p t_1^2 (1 - t_1^2)}{(2 - t_1^2)E'(t_1) - t_1^2 K'(t_1)} \tag{57b}$$

The pressure on the upper surface p_u is given through equation (7) as

$$p_u = \rho V A_q z \left[\frac{2t_1^2 - t^2}{3t_1^2(t_1^2 - t^2)^{1/2}}\right] \tag{58a}$$

$$p_u = \rho V A_p z \left[\frac{t}{3(t_1{}^2 - t^2)^{1/2}} \right] \tag{58b}$$

The pitching results may be combined with the well-known conical solution to obtain damping coefficients about axes other than that through the vertex. The quasi-linear theory may be applied to obtain the appropriate nonsteady results.

The expressions given here agree with those of reference 8 in all pertinent details. Consideration of the symmetric solutions with $n = 2$ with specified pressure on a wing inside the Mach cone is included in the appendix.

Triangular Wings outside Mach Cone

The triangular wing with supersonic leading edges is characterized by the tangent of the angle which the leading edge makes with the flow direction, suitably reduced according to the Prandtl-Glauert transformation and the corresponding value of θ_1

$$t_1 = \sec \theta_0 \tag{59}$$

Taking the right-hand side of the wing so that

$$f_2 \equiv 0$$

$$f_1 = 0, \quad \theta_1 > \theta_0$$

the solution may be expressed in the form

$$f_{1_q} = q \left[\frac{t_1}{(t_1{}^2 - 1)} \delta(\theta_1 - \theta_0) - \frac{t_1(2 - t_1{}^2)}{(t_1{}^2 - 1)^{3/2}} 1(\theta_0 - \theta_1) \right] \tag{60a}$$

$$f_{1_p} = p \left[\frac{t_1^2}{(t_1^2 - 1)} \delta(\theta_1 - \theta_0) - \frac{t_1^2}{(t_1^2 - 1)^{3/2}} 1(\theta_0 - \theta_1) \right] \quad (60b)$$

where δ is the Dirac delta function and 1 is the unit step function. This leads to the expressions for the pressure on the upper right-hand surface of the wing outside the Mach cone

$$p_u = \rho V q \frac{1}{(t_1^2 - 1)^{3/2}} \left[t_1(2 - t_1^2)z - x \right] \quad (61a)$$

$$p_u = \rho V p \frac{t_1^2}{(t_1^2 - 1)^{3/2}}(z - t_1 x) \quad (61b)$$

for the pitching and rolling cases, respectively. Corresponding results hold for the left-hand side of the wing.

For the solutions in the interior of the Mach cone it is most convenient to consider the required form for the functions φ_{yz} and φ_{yx} in the two cases. A consideration of the singularities leads to the results

$$\varphi_{yz} = Rl \left[\frac{q}{i\pi} \left(\log_e \frac{\zeta + t_1}{\zeta - t_1} + \frac{t_1}{\zeta - t_1} + \frac{t_1}{\zeta + t_1} \right) \right] \quad (62a)$$

$$\varphi_{yx} = Rl \left[\frac{p}{i\pi} \left(\log_e \frac{\zeta + t_1}{\zeta - t_1} - \frac{t_1}{\zeta - t_1} - \frac{t_1}{\zeta + t_1} \right) \right] \quad (62b)$$

whence the form of the function $df/d\zeta$ may be obtained

$$f_q{}'(\zeta) = -\frac{q}{\pi}\frac{\zeta}{\sqrt{1-\zeta^2}}\left[\frac{2t_1}{\zeta^2-t_1{}^2}+\frac{t_1}{(\zeta-t_1)^2}+\frac{t_1}{(\zeta+t_1)^2}\right] \qquad (63a)$$

$$f_p{}'(\zeta) = -\frac{p}{\pi}\frac{\zeta^2}{\sqrt{1-\zeta^2}}\left[-\frac{2t_1}{\zeta^2-t_1{}^2}+\frac{t_1}{(\zeta-t_1)^2}+\frac{t_1}{(\zeta+t_1)^2}\right] \qquad (63b)$$

Application of equation (48a) then gives the pressure on the upper surface of the wing

$$p_u = \rho V q \,\frac{1}{\pi(t_1{}^2-1)^{3/2}}\left\{-2t_1\sqrt{t_1{}^2-1}\sqrt{z^2-x^2}+\right.$$

$$\left[t_1(2-t_1{}^2)z-x\right]\cos^{-1}\frac{1-t_1t}{t_1+t}+$$

$$\left.\left[t_1(2-t_1{}^2)+x\right]\cos^{-1}\frac{1+t_1t}{t_1+t}\right\} \qquad (64a)$$

$$p_u = \rho V p\,\frac{t_1{}^2}{\pi(t_1{}^2-1)^{3/2}}\left[(z-t_1x)\cos^{-1}\frac{1-t_1t}{t_1-t}-\right.$$

$$\left.(z+t_1x)\cos^{-1}\frac{1+t_1t}{t_1+t}\right] \qquad (64b)$$

These results check with those of previous investigators, for example, reference 8. It may be readily checked that equations (64) fit equations (61) at $t=1$ and that equations (63) are consistent with equations (60) on the Mach cone.

FAMILY OF SWEPT DELTA WINGS

The methods given above have been applied to the calculation of the wave drag of a family of thin wings at zero incidence. These wings are symmetric laterally and have a symmetric thickness distribution; thus the resulting solution for the velocity potential has the same symmetry and there are no lift forces on the wings. The plan-form shape is a quadrilateral with two of the vertices on the axis, so that the wing appears as a delta wing with sweep.

With the origin at the forward vertex of the wing the leading edge is given by the equation

$$z - \frac{|x|}{t_1} = 0 \tag{65a}$$

where t_1 is a parameter indicating the angle between the leading edge and the flow direction. If the maximum chord on the central axis is c_0 and t_2 is a similar parameter giving the trailing edge direction, the trailing edge is given by the equation

$$c_0 - z - \frac{|x|}{t_2} = 0 \tag{65b}$$

If the wing has a circular-arc profile the upper surface of the wing on the plan form must be given by

$$y_u = \frac{2T}{c_0}\left(z - \frac{|x|}{t_1}\right)\left(c_0 - z - \frac{|x|}{t_2}\right)\psi(|x|) \tag{66}$$

where T is the thickness ratio of the central section and $\psi(0) = 1$. The lower surface of the wing is given simply by the negative of this expression. For a wing with constant thickness ratio the function ψ must be

$$\psi = \left(1 - \frac{t_2 + t_1}{t_2 t_1}\frac{|x|}{c_0}\right)^{-1} \tag{67}$$

Since such a wing cannot be represented by a finite number of homogeneous flows the choice is made here that

$$\psi \equiv 1 \tag{68}$$

The family of wings to be investigated has the thickness distribution given by

$$y_u = \frac{2T}{c_o}\left(z - \frac{|x|}{t_1}\right)\left(c_o - z - \frac{|x|}{t_2}\right) \tag{69}$$

for which the boundary condition is

$$\frac{v_u}{V} = \frac{\partial y_u}{\partial z} = \frac{2T}{c_o}\left(c_o - 2z + \frac{t_2 - t_1}{t_2 t_1}\,|x|\right) \tag{70}$$

It may be noted that such a wing has constant curvature in the axial direction, this being

$$-\frac{\partial^2 y_u}{\partial z^2} = \frac{4T}{c_o} \tag{71}$$

The solution may be considered to consist of three separate parts corresponding to the three terms in the parentheses of equation (70). The first of these is a conical flow of a type much utilized in delta-wing calculations and the solution will not be expressed here. The other two are of the second-degree conical-flow type, one with $\varphi_{yx} = 0$ and

$$\varphi_{yz} = -\frac{4TV}{c_o} \tag{72a}$$

and the other with $\varphi_{yz} = 0$ and

$$\varphi_{yx} = \frac{2TV}{c_o}\frac{t_2 - t_1}{t_2 t_1}\frac{|x|}{x} \tag{72b}$$

The first of these two solutions of the second degree is, for $t_1 > 1$ and the wing lying outside the Mach cone of the vertex, identical with the pitching solution of the preceding section with

$$q = \frac{4TV}{c_o} \tag{73}$$

For the purpose of comparing solutions with t_1 greater than and less than 1 the pressure solution (64a) may be rewritten as

$$p_u = \frac{4T\rho V^2}{c_o} \frac{1}{\pi(t_1^2 - 1)^{3/2}} \left\{ -2t_1\sqrt{t_1^2 - 1} \sqrt{z^2 - x^2} + \right.$$

$$\left[t_1(2 - t_1^2)z - x \right] \tan^{-1} \frac{\sqrt{t_1^2 - 1}\sqrt{1 - t^2}}{1 - t_1 t} +$$

$$\left. \left[t_1(2 - t_1^2)z + x \right] \tan^{-1} \frac{\sqrt{t_1^2 - 1}\sqrt{1 - t^2}}{1 + t_1 t} \right\} \qquad (74a)$$

For the case where $t_1 < 1$ the corresponding solution is immediately obtained from equation (74a)

$$p_u = \frac{4T\rho V^2}{c_o} \frac{1}{\pi(1 - t_1^2)^{3/2}} \left\{ 2t_1\sqrt{1 - t_1^2} \sqrt{z^2 - x^2} - \right.$$

$$\left[t_1(2 - t_1^2)z - x \right] \tanh^{-1} \frac{\sqrt{1 - t_1^2}\sqrt{1 - t^2}}{1 - t_1 t} -$$

$$\left. \left[t_1(2 - t_1^2)z + x \right] \tanh^{-1} \frac{\sqrt{1 - t_1^2}\sqrt{1 - t^2}}{1 + t_1 t} \right\} \qquad (74b)$$

For the second solution the result has some similarities to the rolling case of the preceding section with

$$p = \frac{2TV}{c_o} \frac{t_1 - t_2}{t_1 t_2} \qquad (75)$$

the solutions exterior to the Mach cone being identical on the right-hand side and of opposite sign on the left. In the interior of the Mach cone equation (62b) must be replaced by

$$\varphi_{yx} = R\ell \left[\frac{p}{i\pi} \left(\log_e \frac{\zeta^2}{\zeta^2 - t_1^2} - \frac{t_1}{\zeta - t_1} + \frac{t_1}{\zeta + t_1} \right) \right] \tag{76}$$

the new singularity at $\zeta = 0$ appearing because of the discontinuous behavior of the derivative of $|x|$. Thus equation (63b) is replaced by

$$f'(\zeta) = -\frac{p}{\pi} \frac{\zeta^2}{\sqrt{1 - \zeta^2}} \left[\frac{-2t_1^2}{\zeta(\zeta^2 - t_1^2)} + \frac{t_1}{(\zeta - t_1)^2} - \frac{t_1}{(\zeta + t_1)^2} \right] \tag{77}$$

With this formula and relation (75) the pressure may be expressed for $t_1 > 1$ as

$$p_u = \frac{2T\rho V^2}{c_0} \frac{t_1 - t_2}{t_1 t_2} \frac{1}{\pi(t_1^2 - 1)^{3/2}} \left[-2t_1 \sqrt{t_1^2 - 1} \sqrt{z^2 - x^2} + \right.$$

$$t_1^2(z - t_1 x) \tan^{-1} \frac{\sqrt{t_1^2 - 1} \sqrt{1 - t^2}}{1 - t_1 t} +$$

$$\left. t_1^2(z + t_1 x) \tan^{-1} \frac{\sqrt{t_1^2 - 1} \sqrt{1 - t^2}}{1 + t_1 t} \right] \tag{78a}$$

Correspondingly, for $t_1 < 1$

$$p_u = \frac{2T\rho V^2}{c_0} \frac{t_1 - t_2}{t_1 t_2} \frac{1}{\pi(1 - t_1^2)^{3/2}} \left[2t_1^2 \sqrt{1 - t_1^2} \sqrt{1 - t^2} - \right.$$

$$t_1^2(z - t_1 x) \tanh^{-1} \frac{\sqrt{1 - t_1^2} \sqrt{1 - t^2}}{1 - t_1 t} -$$

$$\left. t_1^2(z + t_1 x) \tanh^{-1} \frac{\sqrt{1 - t_1^2} \sqrt{1 - t^2}}{1 + t_1 t} \right] \tag{78b}$$

The remainder of the problem is to multiply these pressures by the local angle of attack and to add the results of the integrations of these products over the area of the wing. The numerical work involved in this procedure which yields the drag is exceedingly long and tedious and no attempt will be made here to repeat any of these calculations. The analytical results for the drag of this system of bodies are given directly.

The drag given for this family of bodies by strip theory including the $\sqrt{M^2 - 1}$ factor, which is what the drag should approach for very large Mach numbers, may be expressed by

$$c_{D_0} = \frac{8T^2}{3\sqrt{M^2 - 1}} \tag{79}$$

in terms of the drag coefficient. This is less than the two-dimensional value for a circular-arc airfoil by a factor of 1/2 because the thickness ratio is not constant, varying linearly from a zero value at the wing tips to the reference value T at the root. For convenience the drags are all represented as the ratio of the drag coefficient to this reference fictitious value obtained from strip theory.

The results are given below. The new parameter s is introduced,

$$s = t_1/t_2 \tag{80}$$

which is geometric, independent of M for a certain wing. For $1 < t_1 < \infty$ and $1 < |t_2|$,

$$c_D/c_{D_0} = \frac{(1 - s^2)}{\pi(1 - s)^3} \left\{ \left[\frac{2(1 + 3s^2)}{1 - s^2} - \frac{1 - 2s - s^2}{s(t_2^2 - 1)} \right] \left(\frac{t_2 \tan^{-1}\sqrt{t_2^2 - 1}}{\sqrt{t_2^2 - 1}} \right) - \right.$$

$$\left[\frac{2s(3 + s^2)}{1 - s^2} - \frac{1 + 2s - s^2}{(t_1^2 - 1)} \right] \left(\frac{t_1 \tan^{-1}\sqrt{t_1^2 - 1}}{\sqrt{t_1^2 - 1}} \right) - t_1 \left[\frac{2s}{t_1^2 - 1} + \right.$$

$$\left. \left. \frac{2}{s(t_2^2 - 1)} + \frac{(1 - s^2)^2}{s^2(t_1^2 - 1)(t_2^2 - 1)} \right] \right\} \tag{81}$$

This result must be specialized by suitable limiting processes in several special cases:

Case α: where $t_1 = 1$ and $t_2 = 1/s$,

$$C_D/C_{D_o} = \frac{1}{\pi(1 - s)^3}\left[(2 - s + 8s^2 + s^3)\left(\frac{t_2 \tan^{-1}\sqrt{t_2^2 - 1}}{\sqrt{t_2^2 - 1}}\right) + \right.$$

$$\left. \frac{1}{3}(2 - 26s - s^2 - 4s^3 - s^4)\right]. \tag{82}$$

Case β: where $1/t_2 = 0$ and $s = 0$,

$$C_D/C_{D_o} = \left[1 + \frac{t_1 \tan^{-1}\sqrt{t_1^2 - 1}}{\pi(t_1^2 - 1)^{3/2}} - \frac{1}{\pi t_1(t_1^2 - 1)}\right] \tag{83}$$

Case γ: where $t_2 = -t_1$ and $s = -1$ (swept wing of constant chord),

$$C_D/C_{D_o} = \frac{t_1}{\sqrt{t_1^2 - 1}} \tag{84}$$

Case δ: where $t_2 = t_1$ and $s = 1$,

$$C_D/C_{D_o} = \frac{2}{\pi(t_1^2 - 1)^3}\left[(t_1^6 - 2t_1^4 + 10t_1^2 - 4)\left(\frac{t_1 \tan^{-1}\sqrt{t_1^2 - 1}}{\sqrt{t_1^2 - 1}}\right) + \right.$$

$$\left. \frac{1}{3} t_1(3t_1^4 - 28t_1^2 + 10)\right] \tag{85}$$

Case $(\alpha - \beta)$: where $1/t_2 = 0$ and $t_1 = 1$,

$$C_D/C_{D_o} = 1 + \frac{2}{3\pi} = 1.2122 \tag{86}$$

Case $(a - \delta)$: where $t_1 = t_2 = 1$,

$$C_D/C_{D_o} = \frac{368}{105\pi} = 1.1156 \tag{87}$$

These formulas, with the general reversed-flow theorem for drag (reference 1), give the theoretical wave drag completely for the case in which both leading and trailing edges are supersonic. The case in which both edges are subsonic lies outside the scope of the present analysis. For the case in which one edge is subsonic and the other supersonic, without loss of generality as far as the drag is concerned, the leading edge may be required to be the subsonic one.

The drag coefficients for the case $0 < t_1 < 1$ may be represented as the sum of two terms $C_{D_1} + \Delta C_D$, where C_{D_1} is the drag coefficient as calculated by formulas (81) and (83) with the replacement

$$\frac{t_1 \tan^{-1}\sqrt{t_1{}^2 - 1}}{\sqrt{t_1{}^2 - 1}} \longrightarrow \frac{t_1 \tanh^{-1}\sqrt{1 - t_1{}^2}}{\sqrt{1 - t_1{}^2}} \tag{88}$$

and the ΔC_D is an incremental drag term. This incremental term may be expressed for $0 < t_1 < 1$ and $1 < |t_2|$ as

$$
\Delta C_D/C_{D_o} = \frac{1 - s^2}{\pi(1 - s)^3}\left\{ -\left[\frac{2(1 + 3s^2)}{1 - s^2} - \right.\right.
$$

$$
\left. \frac{1 - 2s - s^2}{s(t_2{}^2 - 1)}\right]\left(\frac{t_2 \tan^{-1}\dfrac{\sqrt{t_2{}^2 - 1}\sqrt{1 - t_1{}^2}}{1 + t_1 t_2}}{\sqrt{t_2{}^2 - 1}}\right) -
$$

$$
\left(\frac{t_1}{\sqrt{1 - t_1{}^2}}\right)\left(\frac{1 - s}{s}\right)\left[\frac{s^2(3 + s)}{1 - s^2} + \frac{1 - 2s - s^2}{s(t_2{}^2 - 1)}\right] +
$$

$$
\left[\frac{2t_1 \log_e\left(\frac{1}{t_1}\right)}{\sqrt{1 - t_1{}^2}}\right]\left(\frac{1 + 3s^2}{1 - s^2} + \frac{1}{1 - t_1{}^2}\right)\right\} \tag{89}
$$

For the special cases limiting operations are again necessary:

Case β: where $1/t_2 = 0$ and $s = 0$,

$$\Delta C_D/C_{D_o} = \frac{1}{\pi}\left[-2\,\tan^{-1}\frac{\sqrt{1-t_1^2}}{t_1} - \frac{1}{t_1\sqrt{1-t_1^2}} + \frac{2t_1(2-t_1^2)}{(1-t_1)^{3/2}}\,\log_e\left(\frac{1}{t_1}\right)\right] \quad (90)$$

By the stratagem of introducing terms of the form $|s|$ the two remaining special cases may be expressed together. In these cases the C_{D_1} term is also needed. For

Case ϵ: where $t_2 = 1$ and $t_1 = s$,

and for

Case κ: where $t_2 = -1$ and $t_1 = -s$,

$$C_{D_1}/C_{D_o} = \frac{1}{\pi(1-s)^3}\left[-(1 + 8s - s^2 + 2s^3)\left(\frac{t_1\,\tanh^{-1}\sqrt{1-t_1^2}}{\sqrt{1-t_1^2}}\right) + \right.$$

$$\left. \frac{1}{3|s|}(1 + 4s + s^2 + 26s^3 - 2s^4)\right] \quad (91a)$$

$$\Delta C_D/C_{D_o} = \frac{1}{\pi\sqrt{1-s^2}\,|s|(1-s)^2}\left[-\frac{1}{3}(1 - 2s - s^2)(1-s)^2 - \right.$$

$$\left. s(3 - 2s + 8s^2 + s^3) + \frac{2s^2(1+s)(2+3s^2)\,\log_e\left(\frac{1}{|s|}\right)}{1-s^2}\right] \quad (91b)$$

Although the actual drag coefficient in these last two cases is obtained by adding the two terms, it is convenient for computational purposes to display them separately.

Calculations based on the foregoing formulas have been made for $s = \pm 1$, $\pm \sqrt{2}/2$, $\pm 1/2$, $\pm 1/4$, and 0. Figure 1 shows the geometry of the family of wings under consideration. Figure 2 shows a plot of $1/t_1$ against $1/t_2$ to illustrate the way in which the parameters are related. Figures 3 to 7 give the drag curves, plotted logarithmically against t_1, resulting from the computations. The calculated values of C_D/C_{D_0}, including the incremental term for $t_1 < 1$, are given in table I.

It might be noted that in using these formulas for computation the greatest care must be taken to choose the correct branch of the function $\tan^{-1}$.

APPLICATION TO UNSTEADY-FLOW PROBLEMS

Basic Equations

The velocity potential in unsteady linearized flow, after the application of the Prandtl-Glauert transformation which leads to equation (9), may be expressed

$$\Phi_{xx} + \Phi_{yy} - \Phi_{zz} = \frac{1}{M^2 - 1} \Phi_{tt} + \frac{2M}{M^2 - 1} \Phi_{tz} \qquad (92)$$

in which the time coordinate is defined as a_0 times the time, so that a_0 does not appear in the equation. A new fictitious time variable is introduced

$$\tau = (M^2 - 1)t - Mz \qquad (93)$$

which leads to the equation

$$\Phi_{xx} + \Phi_{yy} - \Phi_{zz} = -\Phi_{\tau\tau} \qquad (94)$$

The pressure is expressed by the relations

$$-\frac{p}{\rho_0 a_0^2} = M\Phi_z + \Phi_t \qquad (95a)$$

$$- \frac{p}{\rho_o a_o} = M\Phi_z - \Phi_\tau \qquad (95b)$$

where the derivatives with respect to z are taken with t constant and with τ constant, respectively.

Quasi-Stationary Theory

The quasi-stationary theory is obtained by assuming that terms involving the second derivative with respect to time may be neglected, or that only the first two terms in a power series in t need be considered. Thus,

$$\Phi = \varphi_0 + t\varphi_1 \qquad (96a)$$

$$\Phi = \varphi_0 + \frac{M}{M^2 - 1} z\varphi_1 + \frac{1}{M^2 - 1} \tau\varphi_1 \qquad (96b)$$

Denoting

$$\varphi = 0(\varphi_y) \qquad (97)$$

as the solution to the problem in terms of the specified boundary values of φ_y on the plan form of interest, there results

$$\Phi = 0\left(\varphi_{o_y}\right) + \frac{M}{M^2 - 1} 0\left(z\varphi_{1_y}\right) + \frac{\tau}{M^2 - 1} 0\left(\varphi_{1_y}\right)$$

$$= 0\left(\varphi_{o_y}\right) + \frac{M}{M^2 - 1}\left[0\left(z\varphi_{1_y}\right) - z0\left(\varphi_{1_y}\right)\right] + t0\left(\varphi_{1_y}\right) \qquad (98)$$

At $t = 0$ the last term may be dropped but the term in $\dfrac{M}{M^2 - 1}$ remains as a contribution due to unsteady-flow effects. This result also may be obtained by taking the first two terms in a frequency expansion of the results according to the harmonic theory (cf. the results of Miles (reference 9)).

In pitching and rolling motion the solution apart from that due to the instantaneous angle of attack may be divided into two parts, one due to the appropriate angular velocity proper which appears in the φ_0 term and one due to the time rate of change of the local angle of attack which appears in the $\dfrac{M}{M^2 - 1}$ term. For steady pitching and rolling motion the second term does not appear. This may be checked by more accurate analyses which take the rotation of the coordinate system into account.

Harmonic Motion

The solution is now assumed to be harmonic of angular frequency ω in the time. A reduced frequency is introduced

$$k = \frac{\omega}{M^2 - 1} \tag{99}$$

and the new potential function is introduced

$$\Phi = e^{ik\tau}\varphi = e^{i(\omega t - Mkz)}\varphi(x,y,z) \tag{100}$$

This function satisfies the equation

$$\varphi_{xx} + \varphi_{yy} - \varphi_{zz} - k^2\varphi = 0 \tag{101}$$

which reduces to equation (9) for $\omega = 0$. The usual boundary condition in terms of φ_y is of the type

$$\varphi_y = e^{iMkz}\left(e^{-i\omega t}\Phi_y\right) \tag{102}$$

The pressure is expressed

$$-\frac{p}{\rho_0 a_0}e^{-i\omega t} = e^{-iMkz}(M\varphi_z - ik\varphi) \tag{103}$$

A solution is now sought for equation (101) of a form similar to that for steady flow

$$\varphi(x_0, y_0, z_0) = h(R_0) \int^{\tau_0} g(\xi, \tau) \, d\tau \tag{104}$$

where R is the hyperbolic distance

$$R = \sqrt{z^2 - x^2 - y^2} \tag{105}$$

and g is considered to be homogeneous of degree n, so that

$$\xi \frac{\partial g}{\partial \xi} = ng \tag{106}$$

A procedure similar to that followed in the steady-state case shows that the assumed solution is valid if h satisfies the equation

$$h_{RR} + \frac{2(n+1)}{R} h_R + k^2 h = 0 \tag{107}$$

for which the appropriate solution is

$$h = {}_0F_1\left(n + \frac{3}{2}; -\frac{k^2 R^2}{4}\right) = \Gamma\left(n + \frac{3}{2}\right) \frac{J_{n+\frac{1}{2}}(kR)}{\left(\frac{1}{2}kR\right)^{n+\frac{1}{2}}} \tag{108}$$

This approach was suggested by the parallel exposition of Germain and Bader (reference 10).

It is still possible, though the practicality might be questioned, to express a solution in terms of a general function of two variables. Noting that h may be expressed as

$$h = \frac{\left(n + \frac{3}{2}\right)}{n! \sqrt{\pi}} \int_0^\pi \cos(kR \sin \theta) \sin^{2n+1} \theta \, d\theta \tag{109}$$

these solutions may be superposed to obtain a single one of the form

$$\varphi(x_0,y_0,z_0) = \int_0^\pi \int^{\tau_0} g(\xi \sin^2\theta, \tau) \, d\tau \cos (kR_0 \sin \theta) \sin \theta \, d\theta \quad (110)$$

where the function g is a new one suitably constructed from the old ones.

Brown University
    Providence, R. I., August 13, 1951

APPENDIX

CONSIDERATION OF ADDITIONAL SOLUTIONS OF SYMMETRIC WINGS

Since the writing of this report Lomax and Heaslet (reference 11) have shown that the symmetric solution with $n = 2$ with specified pressure on a wing inside the Mach cone is not unique. It is of interest to investigate this from the point of view of the theory presented herein. The quantity n is equal to $\kappa + 1$ of Lomax and Heaslet's theory.

In the lift solutions studied in the present investigation, the $f(\zeta)$ functions were of the generalized form given by

$$\frac{df}{d\zeta} = \frac{\zeta^m}{\left(1 - \frac{\zeta^2}{t_0{}^2}\right)^{n+\frac{1}{2}}} = \frac{n!}{\xi^n} g(\xi,\zeta)$$

where m must satisfy the inequality

$$n \lesseqgtr m \lesseqgtr 2n - 1$$

to prevent unallowable singularities on the Mach cone. If m is odd the solution is laterally symmetric; if m is even the solution is laterally antisymmetric. The number of solutions available is in each case equal to the number needed to cover the possible behavior of φ_y on the wing, so that no eigensolutions are indicated.

In the symmetric or thickness case the solutions are of the form

$$\frac{df}{d\zeta} = \frac{i\zeta^m}{\sqrt{1 - \zeta^2}\left(1 - \frac{\zeta^2}{t_0{}^2}\right)^{n+\frac{1}{2}}}$$

where m now satisfies the inequality

$$n \leqq m \leqq 2n$$

In this case for any n there is one solution available in excess of the number needed to cover the possible behavior of φ_z on the wing, thus yielding one eigensolution for each value of n. For n odd the extra solution appears with m odd and is correspondingly laterally antisymmetric. For n even the extra solution appears with m even and is laterally symmetric. In the case n = 2 the nonuniqueness is that found by Lomax and Heaslet and the theory of this report provides a complete check of their result. The characteristic solution still exists without the lateral symmetry as long as both edges are subsonic leading edges.

In the conical case n = 1 the characteristic solution is laterally antisymmetric and has not, to the authors' knowledge, been previously found. For this case the solution is expressed by

$$\frac{df}{d\zeta} = \frac{i\zeta U}{\sqrt{1 - \zeta^2}\left(1 - \dfrac{\zeta^2}{t_o^2}\right)^{3/2}}$$

giving the normal velocity distribution

$$v = Rl\,\frac{\zeta U}{\sqrt{1 - \dfrac{\zeta^2}{t_o^2}}}$$

This gives for the half-thickness distribution of this wing

$$y = x \cosh^{-1} \frac{zt_o}{x}$$

As was to be expected this distribution of thickness is negative on one side. It may be superposed upon the known solution of constant pressure

$$y = \sqrt{z^2 - \frac{x^2}{t_o^2}}$$

to give a family of conical wings satisfying the same boundary conditions.

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TABLE I

CALCULATED VALUES OF c_D/c_{D_0} OR $c_{D_1}/c_{D_0} + \Delta c_D/c_{D_0}$

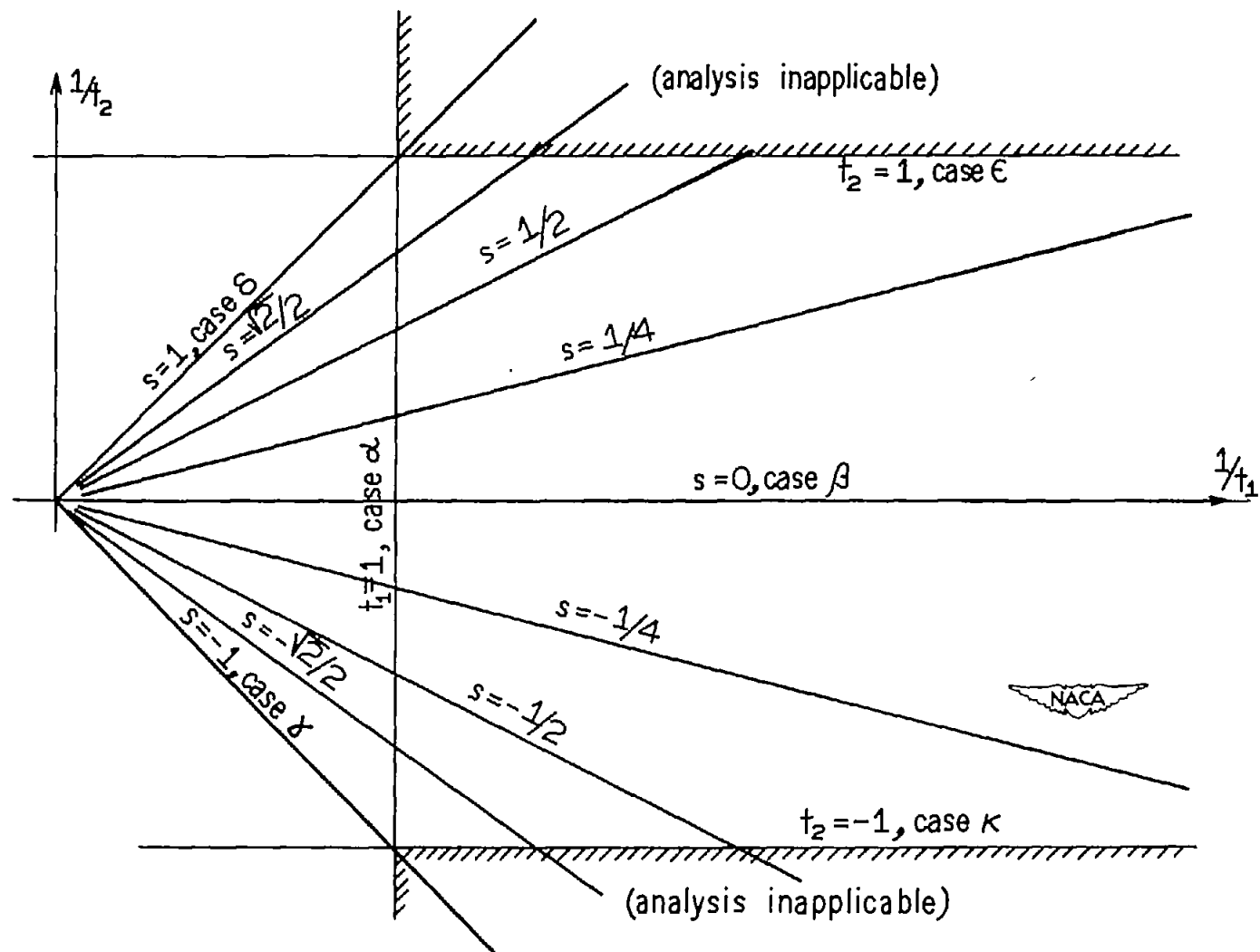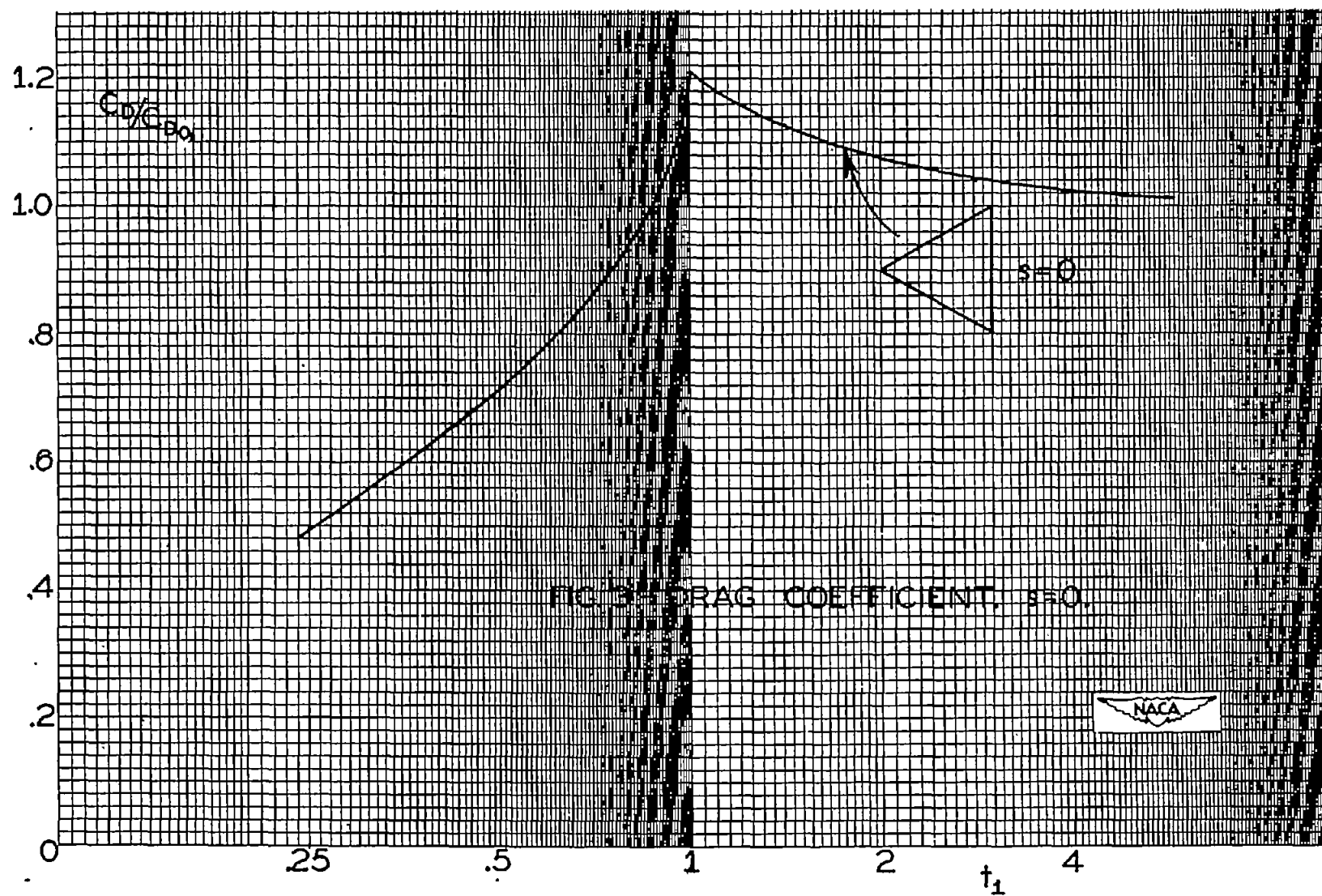| t_2 \ t_1	$4\sqrt{2}$	4	$2\sqrt{2}$	2	$\sqrt{2}$	1	$\sqrt{2}/2$	$1/2$	$\sqrt{2}/4$	$1/4$
1	——	——	——	——	——	1.1156	0.8809	0.7431	——	0.5097
$\sqrt{2}$	——	——	——	——	1.1101	1.1228	.8714	——	0.6131	——
2	——	——	——	1.0899	1.1064	1.1354	——	.7287	——	——
$2\sqrt{2}$	——	——	1.0654	1.0811	1.1073	——	.8643	——	——	——
4	——	1.0434	1.0560	1.0768	——	1.1630	——	——	——	——
$4\sqrt{2}$	1.0268	1.0360	1.0506	——	1.1138	——	——	——	——	——
8	1.0213	1.0311	——	1.0742	——	——	——	——	——	——
∞	1.0127	1.0236	1.0427	1.0753	1.1285	1.2122	.8645	.7188	.5989	.4940
-8	1.0125	1.0229	——	1.0812	——	——	——	——	——	——
$-4\sqrt{2}$	1.0160	1.0254	1.0460	——	1.1552	——	——	——	——	——
-4	——	1.0328	1.0524	1.0941	——	1.3008	——	——	——	——
$-2\sqrt{2}$	——	——	1.0690	1.1120	1.2008	——	.8771	——	——	——
-2	——	——	——	1.1547	1.2619	1.4754	——	.7168	——	——
$-\sqrt{2}$	——	——	——	——	1.4142	1.8035	.9147	——	.5898	——
-1	——	——	——	——	——	∞	.9924	.7301	——	.4820

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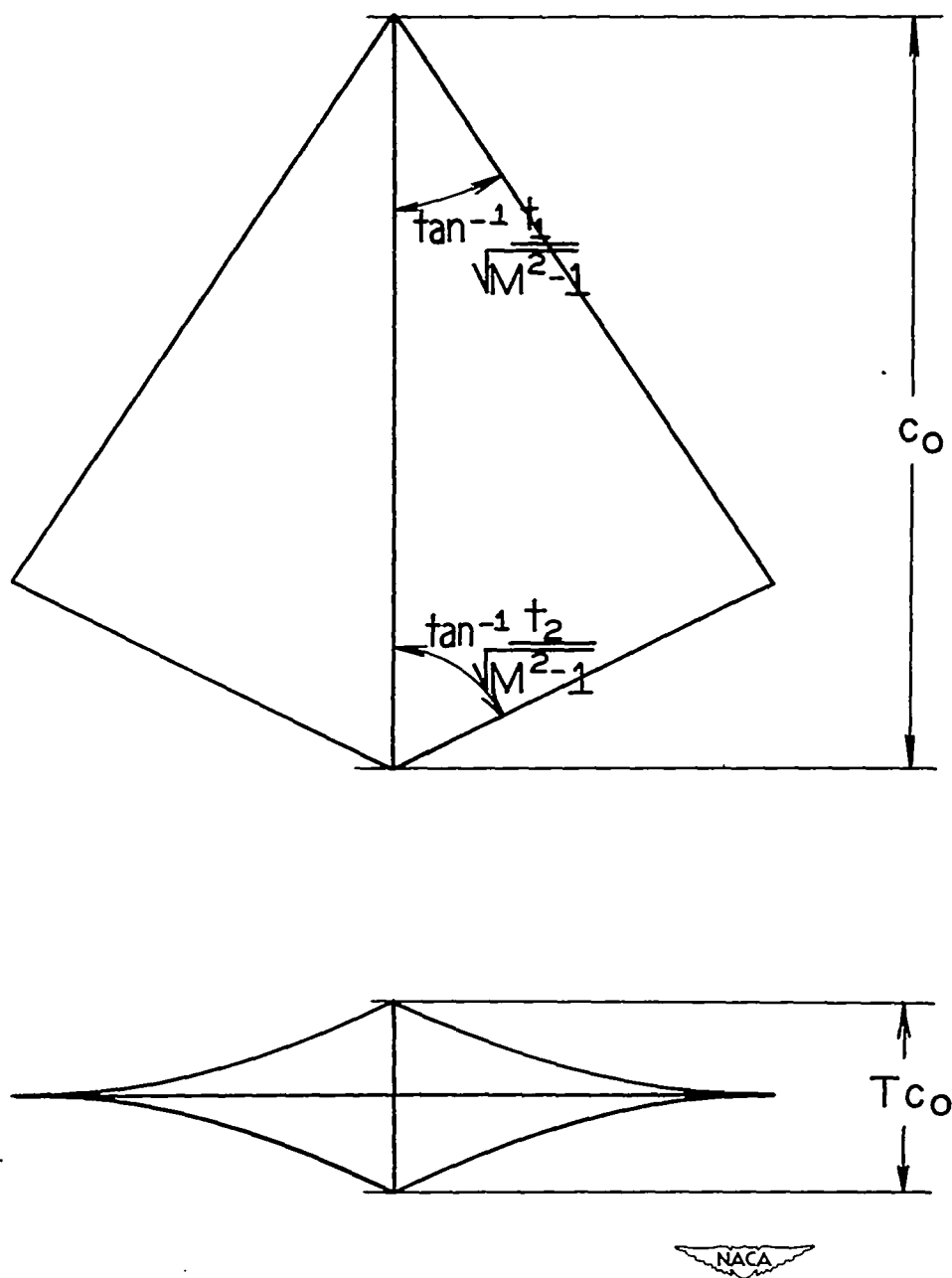
FIGURE 1.—WING GEOMETRY.

FIGURE 2.- PARAMETER CHART.

FIG. 3. DRAG COEFFICIENT, $s=0$.

C_D/C_{D_0}
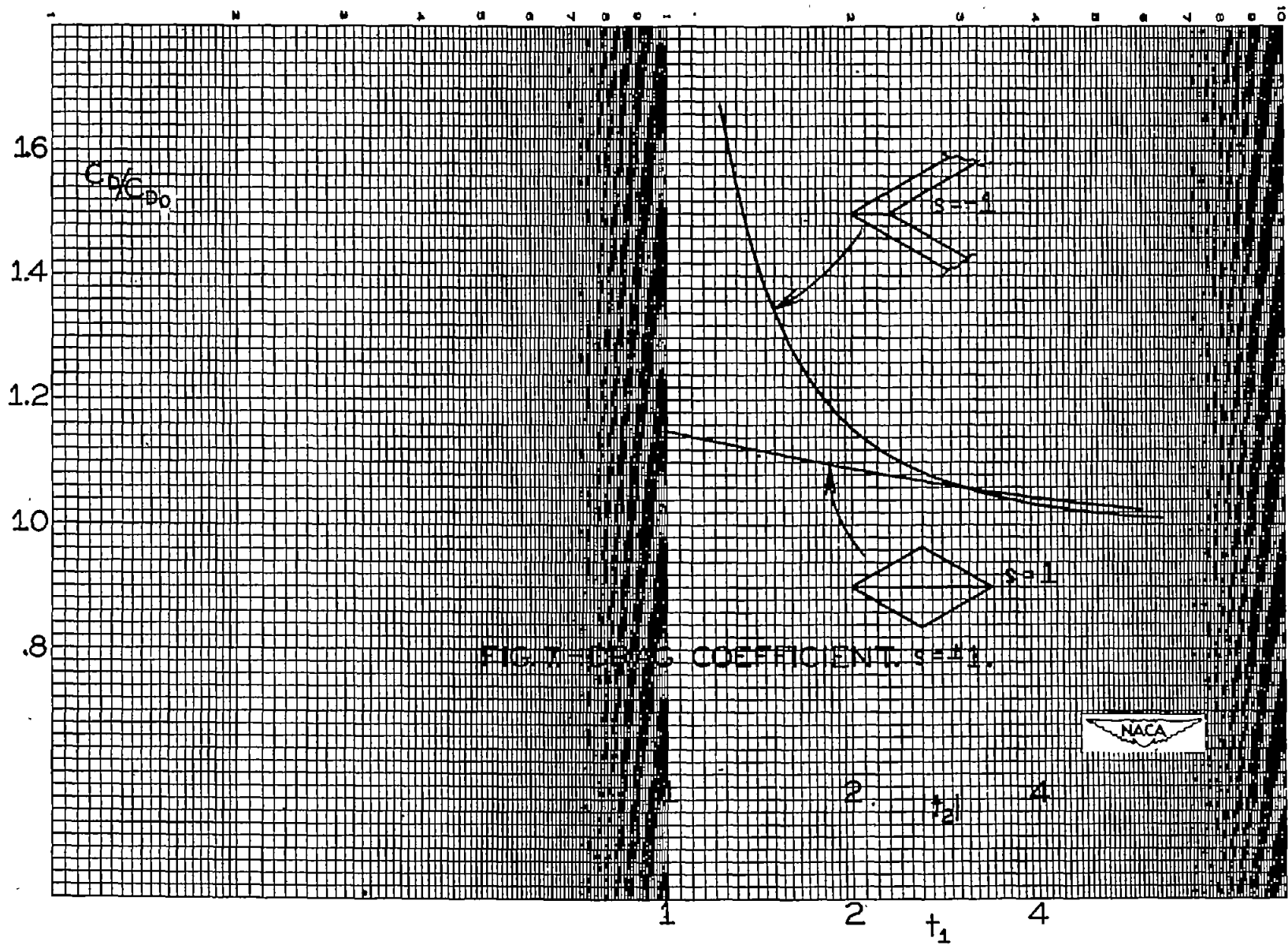
$s=0$

t_1

C_D/C_{D_0}

$s = 1/4$

$s = 1/4$

TOTAL-DRAG COEFFICIENT, $s = 1/4$

t_1

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FIG. DRAG COEFFICIENT, $s = 1/2$.

C_D/C_{D_0}

$s = -1/2$

$s = 1/2$

t_1

FIG. 7.— DRAG COEFFICIENT. S=1.

C_D/C_{D_0}

S=-1

S=1

t_1

t_2

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